

TRANSITION MATRICES FOR SYMMETRIC AND QUASISYMMETRIC HALL-LITTLEWOOD POLYNOMIALS

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ABSTRACT. We introduce explicit combinatorial interpretations for the coefficients in some of the transition matrices between skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x; t)$, Hivert's quasisymmetric Hall-Littlewood polynomials $G_\gamma(x; t)$, and Gessel's fundamental and monomial quasisymmetric functions, $F_\alpha(x)$ and $M_\beta(x)$. More specifically, we provide the following:

- (1) an expansion of the P_λ in terms of the G_γ ;
- (2) expansions of the F_α and the M_β in terms of the G_γ ;
- (3) an expansion of the $P_{\lambda/\mu}$ in terms of the F_α .

The F_α expansion of the $P_{\lambda/\mu}$ is facilitated by introducing the set of *starred tableaux*.

1. INTRODUCTION

The ring of symmetric functions Sym and the ring of quasisymmetric functions QSym both play important roles in algebra and combinatorics. Much of the combinatorial richness arising from these rings stems from their various distinguished bases and the relationships between these bases. The goal of this paper is to present explicit, combinatorial descriptions of several such transition matrices relating to the Hall-Littlewood polynomials. Figure 1 illustrates the bases discussed.

In the top triangle in Figure 1 are included two classical bases for the ring of symmetric functions: the *Schur functions* s_μ and the *monomial symmetric functions* m_ν . The s_μ and m_ν are closely related to a third, one-parameter family of symmetric functions $P_\lambda(x; t)$, known as *Hall-Littlewood polynomials*. More specifically, P_λ equals s_λ at $t = 0$, and it equals m_λ at $t = 1$. The P_λ arose out of a problem studied by P. Hall. Hall had used his eponymous algebra (isomorphic to the algebra of symmetric functions) to encode the structure of finite abelian p -groups. However, at the time there was no known explicit basis of symmetric functions with the same structure constants as that of the natural basis for Hall's algebra. D. E. Littlewood [11] solved this problem in 1961 with his introduction of the $P_\lambda(x; t)$.

The bottom triangle of Figure 1 consists of quasisymmetric analogues of the above bases. In the context of quasisymmetric functions, the *monomial quasisymmetric functions*, M_β , are a very natural analogue of the m_ν . Moreover, there do exist quasisymmetric Schur functions [7]. However, for reasons described in the next paragraph, we anchor the lower-left portion of the bottom triangle in Figure 1 by Gessel's *fundamental quasisymmetric functions*, denoted here by F_α . By defining an action on the Hecke algebra which leaves the quasisymmetric functions invariant, Hivert [8] has constructed the *quasisymmetric Hall-Littlewood polynomials* $G_\gamma(x; t)$.

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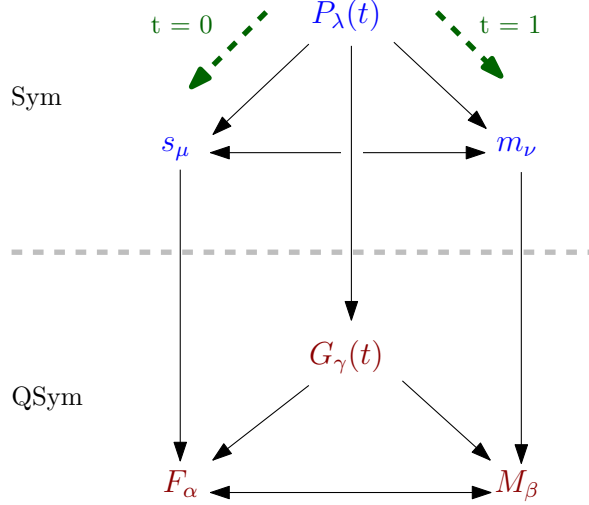


FIGURE 1. Prism of bases and transitions.

Similarly to what happens in the top triangle, specialization of the G_γ at $t = 0$ (which corresponds to the southwest-pointing arrow in Figure 1) yields F_γ , while specialization at $t = 1$ yields M_γ .

We now motivate our choice of the F_α as the desired quasisymmetric analogue of the Schur functions. The Schur functions are the prototypical example of a symmetric function with combinatorial expansions in terms of both a collection of *semistandard objects* (i.e., semistandard Young tableaux) and of *standard objects* (i.e., standard Young tableaux). The first case is that of the classical expansion in terms of monomials weighted by the Kostka numbers. The second expansion (due to Gessel [4]) expresses the Schur functions in terms of fundamental quasisymmetric functions F_α . This expansion, which follows from the technique of *standardization*, is indicated by the vertical line connecting s_λ and F_α in Figure 1. Such standardizations have been used recently to give F -expansions of various symmetric functions including plethysms of Schur functions [13], the modified Macdonald polynomials [5, 6], the Lascoux-Leclerc-Thibon (LLT) polynomials [10], and (conjecturally) the image of a Schur function under the Bergeron-Garsia nabla operator [12].

Given Hivert's construction, the following question arises. Is there an expansion of the P_λ in terms of the G_γ , which would interpolate between the F -expansion of the s_λ at $t = 0$ and the M -expansion of the m_μ at $t = 1$? The main purpose of this paper is to provide such an expansion, and also to provide other change-of-basis matrices between different bases of the Hall algebra and the algebra of quasisymmetric functions, as explained below. In terms of Figure 1, we provide the middle vertical edge as well as the two remaining directed edges in the bottom face.

1.1. G -expansion of the P Basis. In Theorem 23 we give an explicit combinatorial expansion of the Hall-Littlewood polynomials $P_\lambda(x; t)$ in terms of the Hivert quasisymmetric Hall-Littlewood polynomials $G_\gamma(x; t)$. This provides the desired t -interpolation between Gessel's F -expansion of Schur polynomials (i.e., $t = 0$) and the obvious expansion of m_λ 's into M_α 's (i.e., $t = 1$). A key step in our construction is to combine the two expansions described in §1.2 and §1.3.

1.2. F -expansion of the P Basis. One of the main tools for our calculations is the definition of a new class of tableaux, called *starred tableaux*. With these, we give in Theorem 11 a combinatorial expansion of the skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x; t)$ in terms of the fundamental quasisymmetric functions F_α . A minor variation to our method gives a corresponding expansion for the dual Hall-Littlewood polynomials $Q_{\lambda/\mu}$ (see Theorem 16).

1.3. G -expansion of the F and M Bases. In Theorems 17 and 20 we give explicit combinatorial expansions for the F_α and the M_β in terms of the G_γ . These are inverse matrices to those found in [8].

The structure of the paper is as follows. The bases discussed are defined in §2 while the known transition matrices are summarized in §3. The expansions of the Hall-Littlewood polynomials in terms of the F_α and $G_\gamma(x; t)$ are presented in §4 and §5, respectively. Finally, examples of various transition matrices are listed in Appendix 6.

2. REVIEW OF SYMMETRIC AND QUASISYMMETRIC BASES

This section reviews the definitions of the symmetric and quasisymmetric functions appearing in Figure 1. Logically, the precise definitions of the various bases are not needed in this paper, as the expansions found in §4 and §5 are derived from the known transition matrices of §3. However, the material of this section is included for completeness.

2.1. Compositions and Partitions. Given $n \in \mathbb{N}$, a *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of positive integers (called *parts*) with $\alpha_1 + \dots + \alpha_k = n$. Define the *length* $\ell(\alpha)$ to be the number of parts of α , and the *size* $|\alpha|$ to be the sum of its parts. For example, the composition $\alpha = (2, 4, 1)$ has $\ell(\alpha) = 3$ and $|\alpha| = 7$. Let Comp_n be the set of compositions of n , and let Comp be the set of all compositions. A composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \text{Comp}_n$ is called a *partition* of n iff $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. We write Par_n for the set of partitions of n and Par for the set of all partitions.

For $n \in \mathbb{N}^+$, there are 2^{n-1} compositions of n and 2^{n-1} subsets of $[n-1] = \{1, 2, \dots, n-1\}$. One can define natural bijections between these sets of objects as follows. Given $\alpha \in \text{Comp}_n$ as above, let

$$\text{sub}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots, \alpha_1 + \dots + \alpha_{k-1}\} \subseteq [n-1].$$

The inverse bijection sends any subset $T = \{t_1 < t_2 < \dots < t_m\} \subseteq [n-1]$ to

$$\text{comp}(T) = (t_1, t_2 - t_1, t_3 - t_2, \dots, t_m - t_{m-1}, n - t_m) \in \text{Comp}_n.$$

Given $\alpha, \beta \in \text{Comp}_n$, we say β is *finer* than α , denoted $\beta \succeq \alpha$, iff $\text{sub}(\alpha) \subseteq \text{sub}(\beta)$. Informally, β is finer than α if we can chop up some of the parts of α into smaller pieces (without reordering anything) and obtain β . For example, $(1, 1, 1, 1) \succeq (1, 2, 1) \succeq (3, 1) \succeq (4)$.

2.2. Monomial, Schur, and Hall-Littlewood Symmetric Polynomials. Let K be a field of characteristic zero, and let \mathfrak{S}_N denote the symmetric group on N letters. A polynomial $f \in K[x_1, \dots, x_N]$ is called *symmetric* iff

$$f(x_{w(1)}, x_{w(2)}, \dots, x_{w(N)}) = f(x_1, x_2, \dots, x_N) \text{ for all } w \in \mathfrak{S}_N.$$

Write Sym_N for the ring of symmetric polynomials in N variables. For each $n \geq 0$, let Sym_N^n be the subspace of Sym_N consisting of zero and the homogeneous polynomials of degree n . For $N \geq n$, bases of the vector space Sym_N^n are naturally indexed by partitions of n .

Given $\lambda \in \text{Par}_n$ of length $k \leq N$, the *monomial symmetric polynomial* $m_\lambda(x_1, \dots, x_N)$ is the sum of all distinct monomials that can be obtained by permuting subscripts in $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_k^{\lambda_k}$. For $N \geq n$, $\{m_\lambda(x_1, \dots, x_N) : \lambda \in \text{Par}_n\}$ is readily seen to be a basis of Sym_N^n .

Now suppose $N \geq n$ and $\nu \in \text{Par}_n$ is a partition with *distinct* parts. Abusing notation, we append parts of size zero to the end of ν to make ν have length N . The *monomial antisymmetric polynomial indexed by ν in N variables* is

$$(1) \quad a_\nu(x_1, \dots, x_N) = \sum_{w \in \mathfrak{S}_N} \prod_{i=1}^N \text{sgn}(w) x_{w(i)}^{\nu_i} = \det \|x_i^{\nu_j}\|_{1 \leq i, j \leq N}.$$

In particular, letting $\delta_N = (N-1, N-2, \dots, 2, 1, 0)$, $a_{\delta_N}(x_1, \dots, x_N) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$ is the Vandermonde determinant. Given $\lambda \in \text{Par}_n$, the *Schur symmetric polynomial indexed by λ in N variables* is

$$(2) \quad s_\lambda(x_1, \dots, x_N) = \frac{a_{\lambda + \delta_N}(x_1, \dots, x_N)}{a_{\delta_N}(x_1, \dots, x_N)}.$$

It can be shown that this rational function is both a polynomial and symmetric. Moreover $\{s_\lambda : \lambda \in \text{Par}_n\}$ is a basis of Sym_N^n [14, §I.3, p. 40].

For the rest of the paper, let t be an indeterminate, and let K be any field containing $\mathbb{Q}(t)$ as a subfield. Following [14, §III.1, pp. 204–7], we define the *Hall-Littlewood symmetric polynomials* as follows. Fix $\lambda \in \text{Par}_n$ and $N \geq n$. Extend λ to have length N by appending parts of size zero if needed. Define

$$(3) \quad R_\lambda(x_1, \dots, x_N; t) = \frac{\sum_{w \in \mathfrak{S}_N} \text{sgn}(w) x_{w(1)}^{\lambda_1} \cdots x_{w(N)}^{\lambda_N} \prod_{1 \leq i < j \leq N} (x_{w(i)} - tx_{w(j)})}{\prod_{1 \leq i < j \leq N} (x_i - x_j)}.$$

Define $[m]_t = 1 + t + t^2 + \cdots + t^{m-1}$, $[0]_t = 0$, $[m]!_t = \prod_{i=1}^m [i]_t$, and $[0]!_t = 1$. Given that λ has m_0 parts equal to 0, m_1 parts equal to 1, and so on, it can be shown that R_λ is divisible by $[m_0]!_t [m_1]!_t \cdots [m_N]!_t$. We then define the Hall-Littlewood polynomial

$$(4) \quad P_\lambda(x_1, \dots, x_N; t) = \frac{R_\lambda(x_1, \dots, x_N; t)}{[m_0]!_t [m_1]!_t \cdots [m_N]!_t}.$$

It can be shown [14, §III.2, p. 209] that for $N \geq n$, the set $\{P_\lambda(x_1, \dots, x_N; t) : \lambda \in \text{Par}_n\}$ is a basis for Sym_N^n . Moreover, setting $t = 0$ in P_λ gives s_λ , whereas setting $t = 1$ in P_λ gives m_λ . Thus, the Hall-Littlewood basis “interpolates” between the Schur basis and the monomial basis.

One can define Schur polynomials and Hall-Littlewood polynomials more concretely by giving combinatorial descriptions of their expansions in terms of monomial symmetric polynomials. See §3.1 and §3.4 below.

2.3. Monomial, Fundamental, and Hall-Littlewood Quasisymmetric Polynomials. A polynomial $f \in K[x_1, \dots, x_N]$ is called *quasisymmetric* iff for every composition $\alpha = (\alpha_1, \dots, \alpha_k)$ with at most N parts and every $1 \leq i_1 < i_2 < \cdots < i_k \leq N$, the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ have the same coefficient in f . Write QSym_N for the ring of quasisymmetric polynomials in N variables. For each $n \geq 0$, let QSym_N^n be the subspace of QSym_N consisting of zero and the homogeneous polynomials of degree n . For $N \geq n$, bases of the vector space QSym_N^n are naturally indexed by compositions of n . Symmetric polynomials are quasisymmetric, so Sym_N^n is a subspace of QSym_N^n .

For $\alpha \in \text{Comp}_n$ of length $k \leq N$, the *monomial quasisymmetric polynomial* $M_\alpha(x_1, \dots, x_N)$ is the sum of all monomials $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for which $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. For $N \geq n$, $\{M_\alpha(x_1, \dots, x_N) : \alpha \in \text{Comp}_n\}$ is readily seen to be a basis of QSym_N^n .

Next, for $\alpha \in \text{Comp}_n$ with length at most N , define Gessel’s *fundamental quasisymmetric polynomial* [4] by

$$(5) \quad F_\alpha(x_1, \dots, x_N) = \sum_{w \in \mathfrak{S}_N} x_{w_1} x_{w_2} \cdots x_{w_n},$$

where we sum over all subscript sequences $w = w_1 w_2 \cdots w_n$ such that $1 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq N$ and for all $j \in \text{sub}(\alpha)$, $w_j < w_{j+1}$. In other words, strict increases in the subscripts are required in the “breaks” between parts of the composition α . Call sequences w satisfying these conditions α -compatible or $\text{sub}(\alpha)$ -compatible, and write $x^w = x_{w_1} \cdots x_{w_n}$. A routine inclusion-exclusion argument (cf. §3.8 below) shows that for $N \geq n$, $\{F_\alpha(x_1, \dots, x_N) : \alpha \in \text{Comp}_n\}$ is a basis of QSym_N^n . Note that some authors index fundamental quasisymmetric polynomials by pairs n, T where $T \subseteq [n-1]$. Additionally, various letters (F , L , Q , etc.) have been used to denote these polynomials.

As in the symmetric case, we would like to have quasisymmetric Hall-Littlewood polynomials (depending on a parameter t) that interpolate between F_α (when $t = 0$) and M_α (when $t = 1$). We sketch the definition of one such family of polynomials, introduced and studied by Hivert [8]. Quasisymmetric functions arise as the invariants of a certain action of \mathfrak{S}_n on polynomials. From this action, one can define divided difference operators on a degenerate Hecke algebra $H_n(0)$ which can then be lifted to $H_n(q)$. Hivert’s quasisymmetric Hall-Littlewood polynomials thereby arise from a corresponding t -analogue \square_ω of the Weyl symmetrizer. For a composition α of length $k \leq N$, define

$$(6) \quad G_\alpha(x_1, \dots, x_N; t) = \frac{1}{[k]!_t [N-k]!_t} \square_\omega (x_1^{\alpha_1} \cdots x_k^{\alpha_k}).$$

As in the case of symmetric Hall-Littlewood polynomials, there is a more concrete combinatorial definition of G_α giving its expansion into monomials. We discuss this definition in §3.10.

2.4. Non-Commutative Symmetric Functions. Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon [3] have introduced the *algebra of noncommutative symmetric functions* NSym as the free associative algebra generated by the noncommuting indeterminates S_1, S_2, \dots . These S_k are called the *noncommutative complete symmetric functions*. The subalgebra NSym_k is the one generated by S_1, \dots, S_k .

The S_k are the noncommutative analogues of the complete symmetric functions. In fact, the mapping $S_k \mapsto h_k$ gives rise to a surjection from NSym to Sym . Thus, one can view the algebra of noncommutative symmetric functions as a lifting of Sym . (A similar comment applies to NSym_k and Sym_k .)

The S_k can be interpreted in a more concrete way as follows. Consider the infinite alphabet $A = \{a_1 \leq a_2 \leq \cdots\}$. S_k is defined as the formal sum of nondecreasing words in the alphabet A , or, in a way more in tune with the ordinary definition of the h_k , by the generating series

$$(7) \quad \sigma(t) = \sum_{k \geq 0} t^k S_k(A) = (1 - ta_1)^{-1} (1 - ta_2)^{-1} (1 - ta_3)^{-1} \cdots.$$

Another generating set for NSym is the set of *noncommutative elementary symmetric functions* Λ_k , defined as the formal sum of decreasing words in A , or equivalently, by the series

$$(8) \quad \sum_{k \geq 0} t^k \Lambda_k(A) = \cdots (1 + ta_3)(1 + ta_2)(1 + ta_1).$$

The *noncommutative power sums* (of the first kind) Ψ_k are defined by the series

$$(9) \quad \psi(t) = \sum_{k \geq 0} t^{k-1} \Psi_k(A),$$

that satisfies

$$(10) \quad \frac{d}{dt} \sigma(t) = \sigma(t) \psi(t).$$

For a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, one can define $S^\alpha = S_{\alpha_1} \cdots S_{\alpha_k}$. One defines Λ^α and Ψ^α similarly. Each of these families is a basis for NSym. The fact that the bases are indexed by compositions should hint at a connection between the algebras NSym and QSym. This is indeed the case, as they form a pair of *dual Hopf algebras*. One defines $\langle \cdot, \cdot \rangle$ as the bilinear form that makes the bases S^α and M_β dual, namely $\langle S^\alpha, M_\beta \rangle = \delta_{\alpha, \beta}$.

Under this pairing, the basis dual to the fundamental quasisymmetric functions F_β is that formed by the *ribbon Schur functions* R^α , defined as the sum of all words in A which have precisely $\text{sub}(\alpha)$ as a descent set. That is, R^α is the sum of all words $a_{b_1} \cdots a_{b_{|\alpha|}}$ for which $b_i \leq b_{i+1}$ if and only if $i \notin \text{sub}(\alpha)$. For example, $R^{(k)} = S_k$, and $R^{(1^k)} = \Lambda_k$.

As in the symmetric and the quasisymmetric cases, there is a family of *noncommutative Hall-Littlewood polynomials* $S^\alpha(t)$, defined by Novelli, Thibon, and Williams [15], which interpolate between the S^α (when $t = 1$) and a new kind of noncommutative symmetric function Σ_α (when $t = 0$). We discuss these definitions and some of the associated transition matrices in §3.11.

3. REVIEW OF KNOWN TRANSITION MATRICES

In the theory of symmetric and quasisymmetric polynomials, much combinatorial information is encoded in the transition matrices between various bases. Given two bases $B = \{B_\lambda : \lambda \in \text{Par}_n\}$ and $C = \{C_\lambda : \lambda \in \text{Par}_n\}$ of Sym_N^n , the *transition matrix* $M(B, C)$ is the unique matrix (with entries in K and rows and columns indexed by partitions of n) such that

$$(11) \quad B_\lambda = \sum_{\mu \in \text{Par}_n} M(B, C)_{\lambda, \mu} C_\mu.$$

Given a third basis D , it follows readily that $M(B, D) = M(B, C)M(C, D)$ and $M(C, B) = M(B, C)^{-1}$. We define $M(B, C)$ similarly if B and C are bases of QSym_N^n , but here the rows and columns of the matrix are indexed by compositions of n . Finally, if B is a basis of Sym_N^n and C is a basis of QSym_N^n , then $M(B, C)$ is a rectangular matrix expressing each B_λ as a K -linear combination of the C_α 's.

This section gives combinatorial formulas for previously known transition matrices associated to some of the edges in Figure 1. Transitions to the monomial bases offer alternate explicit definitions for Schur polynomials and the various forms of Hall-Littlewood polynomials. Specific examples of these transition matrices appear in Appendix 6.

3.1. $M(s, m)$. The expansion of Schur polynomials into monomials uses semistandard tableaux. For later work, we will also need tableaux of skew shape. Suppose $\lambda, \mu \in \text{Par}$ satisfy $\mu \subseteq \lambda$, i.e., $\mu_i \leq \lambda_i$ for all i . Define the *skew diagram*

$$\lambda/\mu = \{(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ : 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}.$$

We will draw skew diagrams using the English convention where the longest rows are at the top. For $N \in \mathbb{N}^+$, a *semistandard tableau* of shape λ/μ with entries in $[N] = \{1, 2, \dots, N\}$ is a function $T : \lambda/\mu \rightarrow [N]$ that is weakly increasing along rows and strictly increasing down columns. Writing $n = |\lambda/\mu|$, a *standard tableau* of shape λ/μ is a bijection $S : \lambda/\mu \rightarrow [n]$ that is also a semistandard tableau. Let $\text{SSYT}_N(\lambda/\mu)$ be the set of all semistandard tableaux of shape λ/μ with entries in $[N]$, and let $\text{SYT}(\lambda/\mu)$ be the set of all standard tableaux of shape λ/μ . For any $T \in \text{SSYT}_N(\lambda/\mu)$, the *content monomial* x^T is defined to be $\prod_{c \in \lambda/\mu} x_{T(c)}$.

The *skew Schur polynomial* in N variables can now be defined as

$$s_{\lambda/\mu}(x_1, \dots, x_N) = \sum_{T \in \text{SSYT}_N(\lambda/\mu)} x^T.$$

The ordinary Schur polynomial s_λ is obtained by taking $\mu = (0)$ here. For $\lambda, \nu \in \text{Par}_n$ and $N \geq n$, it follows that $M(s, m)_{\lambda, \nu}$ is the Kostka number $K_{\lambda, \nu}$, namely the number of semistandard tableaux of shape λ and content ν .

3.2. $M(s, F)$. The fundamental quasisymmetric expansion of Schur polynomials is a sum over standard tableaux, rather than semistandard tableaux. Given $\lambda \in \text{Par}_n$ and $S \in \text{SYT}(\lambda)$, define the *descent set* $\text{Des}(S)$ to be the set of $k < n$ such that $k + 1$ appears in a lower row of S than k . Define the descent composition $\text{Des}'(S) = \text{comp}(\text{Des}(S))$ to be the composition associated to this subset of $[n - 1]$. Gessel first proved [4] that

$$(12) \quad s_\lambda(x_1, \dots, x_N) = \sum_{S \in \text{SYT}(\lambda)} F_{\text{Des}'(S)}(x_1, \dots, x_N).$$

This formula can be proved bijectively by identifying the individual monomials in $F_{\text{Des}'(S)}$ as the content monomials of semistandard tableaux of shape λ that “standardize” to S . Then $M(s, F)_{\lambda, \alpha}$ is the number of standard tableaux with shape λ and descent set $\text{sub}(\alpha)$.

3.3. $M(m, M)$. For $\lambda \in \text{Par}$, it is immediate that $m_\lambda = \sum M_\alpha$ summed over all compositions α whose parts can be sorted to give the parts of λ . In symbols, $M(m, M)_{\lambda, \alpha}$ is 1 if $\text{sort}(\alpha) = \lambda$ and 0 otherwise.

3.4. $M(P, m)$. Macdonald [14, §III.5, p. 229] gives a formula for the monomial expansion of *skew Hall-Littlewood polynomials* $P_{\lambda/\mu}(x_1, \dots, x_N; t)$, which yields $M(P, m)$ and $M(P, M)$ by taking $\mu = (0)$. We introduce the following combinatorial model for Macdonald’s formula.

Assume λ/μ is a skew shape with $N \geq \ell(\lambda)$. Given $T \in \text{SSYT}_N(\lambda/\mu)$, define the set of *special cells for T* as

$$(13) \quad \text{Sp}(T) = \{(i, j) \in \lambda/\mu : j > 1 \text{ and for all } u \text{ with } (u, j-1) \in \lambda/\mu, T((u, j-1)) \neq T((i, j))\}.$$

Define the *weight* of a special cell (i, j) to be

$$\begin{aligned} \text{wt}((i, j)) &= |\{(u, j-1) \in \lambda/\mu : u \geq i \text{ and } T((u, j-1)) < T((i, j))\}| \\ &\quad + |\{(u, j-1) \in \mu/(0) : u \geq i\}|. \end{aligned}$$

In other words, a cell c with entry $v = T(c)$ is special for T iff c is not in column 1 and there are no v ’s in the column of T just left of c ’s column. In this case, the weight of c is the number of cells weakly below c in the column just left of c that either have entries less than v or are part of the diagram for μ . Now define the set of *starred semistandard tableaux*

$$(14) \quad \text{SSYT}_N^*(\lambda/\mu) = \{(T, E) : T \in \text{SSYT}_N(\lambda/\mu) \text{ and } E \subseteq \text{Sp}(T)\}.$$

A starred tableau $T^* = (T, E)$ has *sign* $\text{sgn}(T^*) = (-1)^{|E|}$, *t-weight* $\text{tstat}(T^*) = \sum_{c \in E} \text{wt}(c)$, *x-weight* $x^{T^*} = x^T$, and *overall weight* $\text{sgn}(T^*)t^{\text{tstat}(T^*)}x^{T^*}$.

For $T \in \text{SSYT}_N(\lambda/\mu)$, Macdonald defines $\psi_T(t) = \prod_{c \in \text{Sp}(T)} (1 - t^{\text{wt}(c)})$. Then Macdonald’s monomial expansion of the skew Hall-Littlewood polynomials is

$$(15) \quad P_{\lambda/\mu}(x_1, \dots, x_N; t) = \sum_{T \in \text{SSYT}_N(\lambda/\mu)} \psi_T(t) x^T.$$

Expanding the product in $\psi_T(t)$ using the distributive law, we get $\sum_{E \subseteq \text{Sp}(T)} \prod_{c \in E} (-t^{\text{wt}(c)})$. Comparing to the overall weight of starred tableaux, we find that

$$(16) \quad P_{\lambda/\mu}(x_1, \dots, x_N; t) = \sum_{T^* \in \text{SSYT}_N^*(\lambda/\mu)} \text{sgn}(T^*) t^{\text{tstat}(T^*)} x^{T^*}.$$

Example 1. Let $\lambda = (8, 6, 5, 4)$, $\mu = (0)$, $N \geq 8$, and

$$(17) \quad T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \underline{2} & 2 & \underline{4} & \underline{5} & 5 \\ \hline 2 & 2 & 3 & 3 & \underline{6} & \underline{8} & & \\ \hline 3 & 3 & \underline{4} & \underline{4} & \underline{7} & & & \\ \hline 5 & 5 & 5 & 5 & & & & \\ \hline \end{array}.$$

In (17), the special cells are indicated by the underlined entries. Specifically,

$$(18) \quad \text{Sp}(T) = \{(1, 4), (1, 6), (1, 7), (2, 5), (2, 6), (3, 3), (3, 5)\}.$$

These special cells have respective weights $1, 1, 1, 3, 2, 1, 2$. So T contributes the term $(1 - t)^4(1 - t^2)^2(1 - t^3)x^T$ to P_λ . A typical starred tableau is $T^* = (T, \{(1, 4), (1, 6), (2, 6)\})$. It can be pictured as follows:

$$(19) \quad T^* = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2^* & 2 & 4^* & 5 & 5 \\ \hline 2 & 2 & 3 & 3 & 6 & 8^* & & \\ \hline 3 & 3 & 4 & 4 & 7 & & & \\ \hline 5 & 5 & 5 & 5 & & & & \\ \hline \end{array}.$$

The overall weight of this object is $(-1)^3 t^{1+1+2} x_1^3 x_2^4 x_3^4 x_4^3 x_5^6 x_6 x_7 x_8 = -t^4 x^T$.

3.5. $M(m, s)$. Egecioglu and Remmel [2] found the following combinatorial formula for the inverse Kostka matrix $M(m, s)$. Fix $\mu \in \text{Par}$. A *special rim-hook* is a sequence of cells in the diagram of μ that begins in the leftmost column and moves up and right through the diagram. The sign of a rim-hook occupying r rows is $(-1)^{r-1}$. A *special rim-hook tableau* S of shape μ is a dissection of the diagram of μ into a disjoint union of special rim-hooks. The *sign* of S , $\text{sgn}(S)$, is the product of the signs of the rim-hooks in it. The *type* of S is the integer partition obtained by listing the lengths of the rim-hooks in S in decreasing order.

Theorem 2. [2, Theorem 1] *For all $\lambda, \mu \in \text{Par}_n$, $M(m, s)_{\lambda, \mu} = \sum \text{sgn}(S)$ summed over all special rim-hook tableaux S of shape μ and type λ .*

3.6. $M(s, P)$. Lascoux and Schützenberger [9] first discovered a combinatorial formula for the “ t -Kostka matrix” $M(s, P)$ involving the famous “charge” statistic. Given a permutation $w = w_1 w_2 \cdots w_n$ of $[n]$, let $\text{IDes}(w)$ be the set of $k < n$ such that $k + 1$ appears to the left of k in w , and let $\text{chg}(w) = \sum_{k \in \text{IDes}(w)} (n - k)$.

Next, let v be a word of partition content (i.e., for all $k \geq 1$, the number of $(k + 1)$ ’s in v is no greater than the number of k ’s). Extract one or more permutations from v as follows. Scan v from left to right marking the first 1, then the first 2 after that, etc., returning to the beginning of v when the right end is reached. Do this until the largest symbol has been marked. Remove the marked symbols from v (in the order they appear) to get the first permutation. Continue to extract permutations in this way until all symbols of v have been used, and let $\text{chg}(v)$ be the sum of the charges of the associated permutations. Finally, given a semistandard tableau T of partition content, let $w(T)$ be the word obtained by reading symbols row by row from top to bottom (i.e., longest row first), reading each row from right to left. Then define $\text{chg}(T) = \text{chg}(w(T))$.

Theorem 3. [9] *For all $\lambda, \mu \in \text{Par}_n$, $M(s, P)_{\lambda, \mu} = \sum t^{\text{chg}(T)}$ summed over all $T \in \text{SSYT}_n(\lambda)$ of content μ .*

3.7. $M(P, s)$. Carbonara [1] gave a combinatorial reformulation of (4) that describes entries of the “inverse t -Kostka matrix” $M(P, s)$ in terms of special tournament matrices. An $n \times n$ *tournament matrix* is a matrix B with entries in $\{0, 1\}$ such that $B_{i,i} = 0$ for all i and, for all $i \neq j$, exactly one of $B_{i,j}$ and $B_{j,i}$ equals 1. Given $\lambda, \mu \in \text{Par}$ having length at most n , we say

the tournament matrix B has *type* λ and *shape* μ iff the sequence $(\lambda_i + \sum_{j=1}^n B_{i,j} : 1 \leq i \leq n)$ is a rearrangement of the sequence $(\mu_i + n - i : 1 \leq i \leq n)$. Such a matrix is *special* (for λ) iff for all $i < j$ with $\lambda_i = \lambda_j$, $B_{ij} = 1$. Let $\Gamma_{\lambda,\mu}^*$ be the set of all $n \times n$ special tournament matrices of type λ and shape μ , where $n = \max(\ell(\lambda), \ell(\mu))$.

We define signs and weights for $B \in \Gamma_{\lambda,\mu}^*$ as follows. Since the entries of $(\mu_i + n - i : 1 \leq i \leq n)$ are distinct, there is a unique $w \in \mathfrak{S}_n$ such that $\lambda_i + \sum_{j=1}^n B_{i,j} = \mu_{w(i)} + n - w(i)$ for $1 \leq i \leq n$. Define $\text{sgn}(B) = \text{sgn}(w)$, where $\text{sgn}(w)$ is the usual sign of the permutation w . Define $\text{wt}(B) = \sum_{i>j} B_{i,j}$, which is the number of nonzero entries of B below the diagonal.

Theorem 4. [1, Theorem 2] *For all $\lambda, \mu \in \text{Par}$, $M(P, s)_{\lambda,\mu} = \sum_{B \in \Gamma_{\lambda,\mu}^*} \text{sgn}(B)(-t)^{\text{wt}(B)}$.*

3.8. $M(F, M)$ and $M(M, F)$. Using (5), one may verify that $F_\alpha = \sum_{\beta \succeq \alpha} M_\beta$. It follows that $M(F, M)_{\alpha,\beta}$ is 1 if $\beta \succeq \alpha$ and 0 otherwise. Now, $M(M, F) = M(F, M)^{-1}$. One can show that $M(M, F)_{\alpha,\beta}$ is $(-1)^{\ell(\beta) - \ell(\alpha)}$ if $\beta \succeq \alpha$ and 0 otherwise. In fact, this is just a matrix formulation of the Möbius inversion formula on the poset of subsets of $[n-1]$ under set inclusion.

3.9. $M(G, F)$. Let $\alpha, \beta \in \text{Comp}_n$ with β finer than α . Say $\ell(\alpha) = k$ and $\ell(\beta) = m$. By definition, there exist indices $0 = i_0 < i_1 < \dots < i_k = m$ such that $\alpha_j = \beta_{i_{j-1}+1} + \dots + \beta_{i_j}$ for $1 \leq j \leq k$. The *refining composition* $\text{Bre}(\beta, \alpha) = (i_1 - i_0, i_2 - i_1, \dots, i_k - i_{k-1})$ records the number of parts of β derived from each part of α . Define $s(\alpha, \beta) = \sum_{j=1}^k j(i_j - i_{j-1} - 1)$. Note that in the notation $\text{Bre}(\beta, \alpha)$ from [8], the finer composition is listed *first*, but in the function s (and g, ξ defined in §5.1), we list the finer composition *second*. This ordering is more convenient when working with transition matrices.

Theorem 5. [8, Theorem 6.6] *For all $N \geq n$ and $\alpha \in \text{Comp}_n$,*

$$(20) \quad G_\alpha(x_1, \dots, x_N; t) = \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} t^{s(\alpha, \beta)} F_\beta(x_1, \dots, x_N).$$

In other words, $M(G, F)_{\alpha,\beta} = (-1)^{\ell(\beta) - \ell(\alpha)} t^{s(\alpha, \beta)}$ if $\beta \succeq \alpha$ and 0 otherwise.

Example 6. Take $\beta = 1221431211$ and $\alpha = 55314$. Then $\text{Bre}(\beta, \alpha) = 32113$ and

$$s(\alpha, \beta) = 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 2 = 14.$$

So $M(G, F)_{\alpha,\beta} = (-1)^5 t^{14}$.

3.10. $M(G, M)$. Using $M(G, M) = M(G, F)M(F, M)$, one can prove the following result giving the monomial expansion of Hivert's quasisymmetric Hall-Littlewood polynomials.

Theorem 7. [8, eq. (105)] *For all $\alpha \in \text{Comp}_n$ and $N \geq n$,*

$$G_\alpha(x_1, \dots, x_N; t) = \sum_{\beta \succeq \alpha} M_\beta(x_1, \dots, x_N; t) \prod_{i=1}^{\ell(\text{Bre}(\beta, \alpha))} (1 - t^i)^{\text{Bre}(\beta, \alpha)_i - 1}.$$

3.11. **Transition Matrices Between Bases in NSym.** Given the duality between the algebras QSym and NSym, and the corresponding duality between important pairs of their bases, some transition matrices between bases in NSym will also give rise to transition matrices between bases in QSym. Here we review some of the basis changes between bases in NSym given by Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon [3], and those involving the expansions of the $S^\alpha(t)$ into these bases, given by Novelli, Thibon, and Williams [15]. Other interesting transitions are found in Tevlin [16].

3.12. $M(S, R)$ **and** $M(R, S)$. In [3], it is shown that the matrix $M(S, R)$ indexed by compositions of n is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\otimes(n-1)}$. This can be restated as follows.

Theorem 8. [3, Proposition 4.13] *For all $\alpha \in \text{Comp}$,*

$$S^\alpha = \sum_{\beta \preceq \alpha} R^\beta \text{ and } R^\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} S^\beta.$$

Note that the first equation says that $M(S, R)_{\alpha, \beta}$ is 1 if and only if $\beta \preceq \alpha$, and the second equation follows from the Möbius inversion formula (cf. §3.8).

3.13. $M(S, \Psi)$ **and** $M(\Psi, S)$. For a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, define $\pi(\alpha)$ by

$$(21) \quad \pi(\alpha) = \alpha_1(\alpha_1 + \alpha_2) \cdots (\alpha_1 + \cdots + \alpha_k).$$

Denote the last part of α by $\text{lp}(\alpha)$, namely, $\text{lp}(\alpha) = \alpha_k$.

For a composition β which is finer than α , let $\beta = (J_1, \dots, J_m)$, where J_i is the composition obtained by considering only the successive parts of β that add to α_i . Now define

$$\pi(\beta, \alpha) = \prod_{i=1}^m \pi(J_i), \text{ and } \text{lp}(\beta, \alpha) = \prod_{i=1}^m \text{lp}(J_i).$$

Theorem 9. [3, Proposition 4.5] *For every composition α , we have*

$$S^\alpha = \sum_{\beta \succeq \alpha} \frac{1}{\pi(\beta, \alpha)} \Psi^\beta \text{ and } \Psi^\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} \text{lp}(\beta, \alpha) S^\beta.$$

3.14. $M(S(t), \Psi)$. We recall some terminology from [15]. For a word w in the alphabet $\{1, 2, \dots\}$, one says w is *packed* if for some positive integer k , the set of letters occurring in w is precisely $\{1, 2, \dots, k\}$. The *word composition* $\text{WC}(w)$ of a packed word w is the composition γ for which $\text{sub}(\gamma)$ gives the positions of the last occurrences of each letter in w . Define the *descent composition* $\text{DC}(w)$ of a word $w = w_1 \cdots w_n$ as the composition γ for which $\text{sub}(\gamma)$ gives precisely the descents of w , i.e., the positions for which $w_i > w_{i+1}$. Finally, for a packed word w , define a *special inversion* as a pair (i, j) where $i < j$, $w_i > w_j$, and $w_j = a$ is the rightmost occurrence of the letter a in w . Let $\text{sinv}(w)$ denote the number of special inversions in w .

For example, let $w = 311232$. We have $\text{WC}(w) = (3, 2, 1)$ since the last 1 is in position 3, the last 2 is in position 6, and the last 3 is in position 5, hence $\text{sub}(3, 2, 1) = \{3, 5, 6\}$. Furthermore, $\text{DC}(w) = (1, 4, 1)$ since the descents of w occur at the positions $\{1, 5\}$. Finally, since the special inversions in w are the pairs $(1, 3), (1, 6), (5, 6)$, then $\text{sinv}(w) = 3$.

Theorem 10. [15, Proposition 3.2] *Let α and β be two compositions, and let $W(\alpha, \beta)$ be the set of packed words w such that $\text{WC}(w) = \alpha$ and $\text{WC}(w) \succeq \beta$. Then*

$$M(S(t), \Psi)_{\alpha, \beta} = \sum_{w \in W(\beta, \alpha)} t^{\text{sinv}(w)}.$$

3.15. $M(S(t), S)$ **and** $M(S(t), R)$. By multiplying $M(S(t), \Psi)$, $M(\Psi, S)$, and $M(S, R)$, one can obtain the matrices $M(S(t), S)$ and $M(S(t), R)$. By duality, these last two are precisely the transposes of the matrices $M(M, G)$ and $M(F, G)$. However, we are unaware of any interpretations of the last two as single matrices, aside from the ones in we provide in §5.1 and §5.2.

4. F -EXPANSION OF SKEW HALL-LITTLEWOOD POLYNOMIALS

4.1. Expansion of $P_{\lambda/\mu}$. Recall from §3.4 the combinatorial formula (16) for the monomial expansion of the skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x_1, \dots, x_N; t)$. This section converts this formula to an expansion of these polynomials in terms of the fundamental quasisymmetric basis. In particular, this provides a combinatorial interpretation for the entries of $M(P, F)$. We would like to remark that one can also obtain $M(P, F)$ by multiplying the known matrices $M(P, s)$ and $M(s, F)$. However, this produces a quite complicated interpretation for the coefficients in $M(P, F)$ as signed combinations of standard tableaux and special tournaments. The new interpretation developed below is much simpler.

To state our result, we need a few more definitions. Given a skew diagram λ/μ with n cells, let $\text{SYT}^*(\lambda/\mu)$ be the set of starred standard tableaux $S^* = (S, E)$ such that S is a standard tableau of shape λ/μ . In this case, observe that $\text{Sp}(S)$ consists of all cells in the diagram not in column 1. So E can be an arbitrary subset of cells of λ/μ not in column 1. Define the *ascent set* of S^* , denoted $\text{Asc}(S^*)$, to be the set of all $k < n$ such that either (a) $k + 1$ appears in S in a lower row than k , or (b) there exist u, i, j with $S((u, j - 1)) = k$, $S((i, j)) = k + 1$, and $(i, j) \in E$. The second alternative says that $k + 1$ appears in a cell of E located in the next column after the column containing k . Define $\text{Asc}'(S^*) = \text{comp}(\text{Asc}(S^*))$ to be the associated composition.

Theorem 11. *For all skew shapes λ/μ with $n \leq N$ cells,*

$$P_{\lambda/\mu}(x_1, \dots, x_N; t) = \sum_{S^* \in \text{SYT}^*(\lambda/\mu)} \text{sgn}(S^*) t^{\text{tstat}(S^*)} F_{\text{Asc}'(S^*)}(x_1, \dots, x_N).$$

Proof. Let Y be the set of pairs (S^*, w) where $S^* = (S, E) \in \text{SYT}^*(\lambda/\mu)$ and w is an $\text{Asc}(S^*)$ -compatible word of length n (see §2.3). The overall weight of $(S^*, w) \in Y$ is $\text{sgn}(S^*) t^{\text{tstat}(S^*)} x^w$. Keeping in mind the definition of F_α , we see that the generating function for the weighted set Y is

$$(22) \quad \sum_{y \in Y} \text{wt}(y) = \sum_{S^* \in \text{SYT}^*(\lambda/\mu)} \text{sgn}(S^*) t^{\text{tstat}(S^*)} F_{\text{Asc}'(S^*)}(x_1, \dots, x_N).$$

Comparing (16) and (22), the theorem will be proved if we can construct a sign-preserving, weight-preserving bijection $f : \text{SSYT}^*(\lambda/\mu) \rightarrow Y$.

Example 12. Let $\lambda = (6, 5, 2, 1, 1)$, and consider the starred standard tableau

$$(23) \quad S^* = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 8^* & 9 & 15^* \\ \hline 4 & 6 & 7^* & 13^* & 14 & \\ \hline 5 & 12 & & & & \\ \hline 10 & & & & & \\ \hline 11 & & & & & \\ \hline \end{array}.$$

For this object, $E = \{(1, 4), (1, 6), (2, 3), (2, 4)\}$, $\text{Asc}(S^*) = \{3, 4, 6, 7, 9, 10, 14\}$, and $\text{Asc}'(S^*) = 31212141$. A typical object in Y is (S^*, w) where $w = 111 < 2 < 33 < 4 < 55 < 6 < 7888 < 9$. The marked ascents in w are mandated for w to be $\text{Asc}(S^*)$ -compatible, and there is an extra ascent between $w_{11} = 7$ and $w_{12} = 8$. The overall weight of (S^*, w) is $(-1)^{4+2+2+1+1} x_1^3 x_2^2 x_3^2 x_4^2 x_5^3 x_6 x_7 x_8^3 x_9 = +t^6 x^w$.

Continuing the proof, we must define a “standardization map” $f : \text{SSYT}^*(\lambda/\mu) \rightarrow Y$ and an “unstandardization map” $g : Y \rightarrow \text{SSYT}^*(\lambda/\mu)$ such that f preserves signs and weights, $f \circ g = \text{id}_Y$, and $g \circ f = \text{id}_{\text{SSYT}^*(\lambda/\mu)}$. Given a semistandard tableau $T \in \text{SSYT}_N(\lambda/\mu)$, recall that there is a standard tableau $\text{stdz}(T)$, called the *standardization* of T , defined as follows. Suppose

the entries in T consist of m_1 ones, m_2 twos, etc. Define $M_0 = 0$ and $M_i = m_1 + m_2 + \cdots + m_i$ for $1 \leq i \leq N$. We obtain $\text{stdz}(T)$ from T by replacing the m_i occurrences of i in T , from left to right, by the integers $M_{i-1} + 1, M_{i-1} + 2, \dots, M_i$. Now, given $T^* = (T, E) \in \text{SSYT}^*(\lambda/\mu)$, define $f(T^*) = (S^*, w) = ((\text{stdz}(T), E), w)$, where w consists of the symbols in T in increasing order.

We must check that $f(T^*) \in Y$ for all $T^* = (T, E) \in \text{SSYT}^*(\lambda/\mu)$. First, $S = \text{stdz}(T)$ is a standard tableau of shape λ/μ . Second, since $E \subseteq \text{Sp}(T)$, every cell of E is not in column 1, and so $E \subseteq \text{Sp}(S)$. Third, we claim w is an $\text{Asc}(S^*)$ -compatible sequence. To check this, assume $k < n$ and $w_k = w_{k+1} = v$. By definition of standardization, the unique occurrences of k and $k+1$ in S were used to relabel cells that both originally contained occurrences of v in T . Since the horizontal strip of cells in T containing v gets relabeled from left to right, $k+1$ cannot appear in S in a lower row than k . So condition (a) in the definition of $\text{Asc}(S^*)$ does not hold for k . Can condition (b) hold for k ? If so, there are u, i, j with $S((u, j-1)) = k$ and $S((i, j)) = k+1$ and $(i, j) \in E$. But then $T((u, j-1)) = v = T((i, j))$ means that (i, j) is not a special cell for T , which contradicts $E \subseteq \text{Sp}(T)$. So we conclude that $k \notin \text{Asc}(S^*)$, proving the claim.

With notation as above, observe that $x^{T^*} = x^T = x^w$, so that f preserves the x -weight. Since applying f does not change E , f preserves signs. Finally, suppose $(i, j) \in E$ with $T((i, j)) = v$. Since v cannot appear in column $j-1$ of T , it follows from the definition of standardization that the cells contributing to $\text{wt}((i, j))$ in the computation of $\text{tstat}(T^*)$ are precisely the cells contributing to $\text{wt}((i, j))$ in the computation of $\text{tstat}(S^*)$. So, f preserves the t -weight.

All that remains is to define the two-sided inverse g for f . Given $(S^*, w) = ((S, E), w) \in Y$, suppose w consists of m_1 ones followed by m_2 twos, etc. With M_i defined as before, let $T : \lambda/\mu \rightarrow [N]$ be obtained from S by replacing all symbols $M_{i-1} + 1, \dots, M_i$ by i 's, for $1 \leq i \leq N$. Then set $g((S^*, w)) = (T, E)$. We must check that (T, E) lies in the codomain X . Note that w is $\text{Asc}(S^*)$ -compatible, so $w_k = w_{k+1}$ implies that conditions (a) and (b) in the definition of ascent set are both false for this k . The falsehood of condition (a) ensures that T will be a semistandard tableau. On the other hand, the falsehood of condition (b) guarantees that every cell of E is special for T (not just special for S). So $(T, E) \in X$ as needed.

Knowing that f and g do map into their stated codomains, it is now immediate that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_{\text{SSYT}^*(\lambda/\mu)}$. (This verification does not involve the sets E or the ascent sets; one only needs the fact that the usual standardization of a semistandard tableau is reversible if you know the content word w). So the proof is complete. \square

Example 13. Applying f to the starred semistandard tableau T^* from Example 1 gives $f(T^*) = (S^*, w)$, where

$$(24) \quad S^* = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6^* & 7 & 14^* & 19 & 20 \\ \hline 4 & 5 & 10 & 11 & 21 & 23^* & & \\ \hline 8 & 9 & 12 & 13 & 22 & & & \\ \hline 15 & 16 & 17 & 18 & & & & \\ \hline \end{array}$$

and $w = 11122223333444555555678$. Observe that w is $\text{Asc}(S^*)$ -compatible, since $\text{Asc}(S^*) = \{3, 7, 11, 14, 20, 21, 22\}$. Moreover, $g((S^*, w)) = T^*$.

Example 14. Applying g to the object $(S^*, w) \in Y$ from Example 12 gives the starred semistandard tableau

$$(25) \quad T^* = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 5^* & 5 & 9^* \\ \hline 2 & 3 & 4^* & 8^* & 8 & \\ \hline 3 & 8 & & & & \\ \hline 6 & & & & & \\ \hline 7 & & & & & \\ \hline \end{array}.$$

Note that T is semistandard and E does consist of special cells for T . Moreover, $f(T^*) = (S^*, w)$.

Example 15. Using Theorem 11, we can make the following calculation. Each term corresponds to the starred semistandard tableau shown below it:

$$P_{21}(t) = F_{21} - tF_{111} + F_{12} - t^2F_{111}.$$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2^* \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3^* \\ \hline 2 & \\ \hline \end{array}$$

4.2. Expansion of $Q_{\lambda/\mu}$. We recall the definition of the skew Hall-Littlewood polynomials $Q_{\lambda/\mu}(x_1, \dots, x_N; t)$ [14, §III.5]. For $r \in \mathbb{N}^+$, let $\phi_r(t) = (1-t)(1-t^2) \cdots (1-t^r)$. For a partition λ with $m_i(\lambda)$ parts equal to i for each i , define $b_\lambda(t) = \prod_{i \geq 1} \phi_{m_i(\lambda)}(t)$. For partitions $\mu \subseteq \lambda$, define $b_{\lambda/\mu}(t) = b_\lambda(t)/b_\mu(t)$. Finally, define

$$(26) \quad Q_{\lambda/\mu}(x_1, \dots, x_N; t) = b_{\lambda/\mu}(t)P_{\lambda/\mu}(x_1, \dots, x_N; t).$$

From this, Theorem 11 immediately gives

$$Q_{\lambda/\mu}(x_1, \dots, x_N; t) = \sum_{S^* \in \text{SYT}^*(\lambda/\mu)} \text{sgn}(S^*) t^{\text{tstat}(S^*)} b_{\lambda/\mu}(t) F_{\text{Asc}'(S^*)}(x_1, \dots, x_N).$$

On the other hand, Macdonald [14, §III.5, pp. 227–229] gives the following monomial expansion of $Q_{\lambda/\mu}$. Suppose $\nu \subseteq \rho$ are partitions such that $\theta = \rho/\nu$ is a horizontal strip. Let θ'_i be the number of cells of θ in column i , so $\theta'_i \in \{0, 1\}$ since θ is a horizontal strip. Define

$$(27) \quad \phi_{\rho/\nu}(t) = \prod_{i \in I} (1 - t^{m_i(\rho)})$$

where $I = \{i \geq 1 : \theta'_i = 1 \text{ and } \theta'_{i+1} = 0\}$. For $T \in \text{SSYT}_N(\lambda/\mu)$, view T as a nested sequence of partitions $\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)} = \lambda$, where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip consisting of the cells in T with entry i , and let $\phi_T(t) = \prod_{i=1}^N \phi_{\lambda^{(i)}/\lambda^{(i-1)}}(t)$. Macdonald's expansion for $Q_{\lambda/\mu}$ is

$$(28) \quad Q_{\lambda/\mu}(x_1, \dots, x_N; t) = \sum_{T \in \text{SSYT}(\lambda/\mu)} \phi_T(t) x^T.$$

By imitating the proof we gave in §4.1, we can use (28) to derive an alternative fundamental quasisymmetric expansion for $Q_{\lambda/\mu}$ (and hence also for $P_{\lambda/\mu}$).

We merely sketch the necessary changes in the first formula and its proof. For a tableau $T \in \text{SSYT}_N(\lambda/\mu)$, define the set of Q -special cells for T by

$$(29) \quad \text{QSp}(T) = \{(i, j) \in \lambda/\mu : \text{for all } u \in \lambda/\mu, T((u, j+1)) \neq T(i, j)\}.$$

So, a cell c is Q -special for T iff the entry of T in c does not also appear in the column immediately to the right of c . The *weight* of a Q -special cell (i, j) is now defined to be

$$(30) \quad \text{wt}((i, j)) = m_j(\lambda^{(T(i, j))}) = i - |\{u : (u, j+1) \in \mu/(0) \text{ or } T((u, j+1)) \leq T(i, j)\}|.$$

Now let $\text{SSYT}_N^{*Q}(\lambda/\mu) = \{(T, E) : T \in \text{SSYT}_N(\lambda/\mu) \text{ and } E \subseteq \text{QSp}(T)\}$ be the set of “ Q -starred” semistandard tableaux. Since $\phi_T(t) = \prod_{c \in \text{QSp}(T)} (1 - t^{\text{wt}(c)})$, it follows as before that

$$(31) \quad Q_{\lambda/\mu}(x_1, \dots, x_N; t) = \sum_{T^* \in \text{SSYT}_N^{*Q}(\lambda/\mu)} \text{wt}(T^*).$$

Define Q -starred standard tableaux in the obvious way. For $S \in \text{SYT}(\lambda/\mu)$, observe that *every* cell in S is Q -special for S . Given $S^* = (S, E)$ with $E \subseteq \lambda/\mu$ and $k < n = |\lambda/\mu|$, define $k \in \text{QAsc}(S^*)$ iff either (a) $k+1$ appears in S in a lower row than k ; or (b) $k \in E$ and $k+1$ appears in S in the next column after k 's column. Let $\text{QAsc}'(S^*) = \text{comp}(\text{QAsc}(S^*))$. Define Y as before, replacing $\text{Asc}(S^*)$ by $\text{QAsc}(S^*)$. One should now check that the proof given above adapts to the present situation without difficulty. We therefore have the following result.

Theorem 16. *For all skew shapes λ/μ with $n \leq N$ cells,*

$$Q_{\lambda/\mu}(x_1, \dots, x_N; t) = \sum_{S^* \in \text{SYT}^{*Q}(\lambda/\mu)} \text{sgn}(S^*) t^{\text{tstat}(S^*)} F_{\text{QAsc}'(S^*)}(x_1, \dots, x_N).$$

Dividing through by $b_{\lambda/\mu}(t)$ gives an analogous F -expansion for $P_{\lambda/\mu}$.

5. NEW TRANSITION MATRICES

This section discusses combinatorial formulas for the transition matrices $M(F, G)$, $M(M, G)$, and $M(P, G)$.

5.1. $M(F, G)$. Let $\alpha, \beta \in \text{Comp}_n$ with β finer than α . Define $\xi_{\alpha, \beta}(j)$ to be j if β_j and β_{j+1} are formed from the same part of α and 0 otherwise. Set $g(\alpha, \beta) = \sum_{j=1}^{\ell(\beta)-1} \xi_{\alpha, \beta}(j)$.

Theorem 17. *For all $\alpha, \beta \in \text{Comp}_n$,*

$$M(F, G)_{\alpha, \beta} = \begin{cases} t^{g(\alpha, \beta)}, & \text{if } \beta \succeq \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $M(F, G)$ is the unique matrix such that $M(G, F)M(F, G) = I$, it is enough (by Theorem 5) to show that for all compositions $\beta \succeq \alpha$,

$$(32) \quad \sum_{\gamma: \beta \succeq \gamma \succeq \alpha} (-1)^{\ell(\gamma) - \ell(\alpha)} t^{s(\alpha, \gamma)} t^{g(\gamma, \beta)} = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that $s(\alpha, \gamma) = \sum_{j=1}^{\ell(\alpha)} j(\text{Bre}(\gamma, \alpha)_j - 1)$. If $\alpha = \beta$ then there is a single term that is readily seen to equal 1. So suppose $\beta \succ \alpha$. We define a sign-reversing involution $\gamma \mapsto \gamma'$ on the set of compositions γ with $\beta \succeq \gamma \succeq \alpha$, as follows. Let j be as small as possible such that $\text{Bre}(\beta, \gamma)_j > 1$ or $\text{Bre}(\gamma, \alpha)_j > 1$. Equivalently, this is the smallest j such that not all of α_j , β_j and γ_j are equal. We know that such a j must exist since $\alpha \neq \beta$.

If $\gamma_j = \beta_j$ (and hence $\beta_j < \alpha_j$), then set

$$(33) \quad \gamma' = (\gamma_1, \dots, \gamma_{j-1}, \gamma_j + \gamma_{j+1}, \gamma_{j+2}, \dots, \gamma_{\ell(\gamma)}).$$

Otherwise, let

$$(34) \quad \gamma' = (\gamma_1, \dots, \gamma_{j-1}, \beta_j, \gamma_j - \beta_j, \gamma_{j+1}, \dots, \gamma_{\ell(\gamma)}).$$

If γ' is defined according to (33), then $s(\alpha, \gamma') = s(\alpha, \gamma) - j$ and $g(\gamma', \beta) = g(\gamma, \beta) + j$. On the other hand, if γ' is defined according to (34), then $s(\alpha, \gamma') = s(\alpha, \gamma) + j$ and $g(\gamma', \beta) = g(\gamma, \beta) - j$. It follows immediately that the map $\gamma \mapsto \gamma'$ is a sign-reversing, t -weight preserving involution on the set of γ for which $\beta \succeq \gamma \succeq \alpha$. Hence, the sum in (32) is zero as desired. \square

Example 18. Let $\alpha = 212135$, $\gamma = 2121323$ and $\beta = 212112212$. Note that $\text{Bre}(\gamma, \alpha) = 111112$ and $\text{Bre}(\beta, \gamma) = 1111212$. It follows that $s(\alpha, \gamma) = 6 \cdot (2 - 1) = 6$ and $g(\gamma, \beta) = 5 + 8 = 13$. The smallest j for which the parts are not all equal is $j = 5$. The parts γ_5 and β_5 are not equal, so γ' gets defined according to (34): $\gamma' = 21211223$. Hence $\text{Bre}(\gamma', \alpha) = 111122$ and $\text{Bre}(\beta, \gamma') = 1111112$. So $s(\alpha, \gamma') = 5 \cdot 1 + 6 \cdot 1 = 11$ and $g(\gamma', \beta) = 8$.

Example 19. Using Theorem 17, we calculate $F_3 = G_3 + tG_{21} + tG_{12} + t^3G_{111}$, $F_{21} = G_{21} + tG_{111}$, $F_{12} = G_{12} + t^2G_{111}$, and $F_{111} = G_{111}$.

5.2. $M(M, G)$.

Theorem 20. For all $\alpha, \beta \in \text{Comp}_n$ with $\beta \succeq \alpha$,

$$M(M, G)_{\alpha, \beta} = (-1)^{\ell(\beta) - \ell(\alpha)} \prod_{j: \xi_{\alpha, \beta}(j) = j} (1 - t^j).$$

For other α, β , $M(M, G)_{\alpha, \beta} = 0$.

Proof. Fix $\alpha, \beta \in \text{Comp}_n$. Since $M(M, G) = M(M, F)M(F, G)$,

$$\begin{aligned} M(M, G)_{\alpha, \beta} &= \sum_{\gamma \in \text{Comp}_n} M(M, F)_{\alpha, \gamma} M(F, G)_{\gamma, \beta} \\ &= \sum_{\gamma: \beta \succeq \gamma \succeq \alpha} (-1)^{\ell(\gamma) - \ell(\alpha)} t^{g(\gamma, \beta)} \\ &= (-1)^{\ell(\beta) - \ell(\alpha)} \sum_{\gamma: \beta \succeq \gamma \succeq \alpha} (-1)^{\ell(\gamma) - \ell(\beta)} t^{g(\gamma, \beta)}. \end{aligned}$$

Any composition γ in the above sum is completely determined by specifying the values of $\xi_{\gamma, \beta}(j)$ for those j such that $\xi_{\alpha, \beta}(j) = j$. This follows since $\beta \succeq \gamma \succeq \alpha$ and $\xi_{\alpha, \beta}(j) = 0$ force $\xi_{\gamma, \beta}(j) = 0$. On the other hand, for each j with $\xi_{\alpha, \beta}(j) = j$, we can always choose $\xi_{\gamma, \beta}(j)$ to be either 0 or j when building γ . Every time we choose to set $\xi_{\gamma, \beta}(j) = j$, we are increasing the length difference between β and γ by 1. Additionally, we are increasing the t -weight by j . Since all of these choices are independent, we can write

$$\sum_{\gamma: \beta \succeq \gamma \succeq \alpha} (-1)^{\ell(\gamma) - \ell(\beta)} t^{g(\gamma, \beta)} = \prod_{j: \xi_{\alpha, \beta}(j) = j} (1 - t^j).$$

The theorem follows. \square

Example 21. Consider $\alpha = 22$ and $\beta = 1111$. Then $\xi_{\alpha, \beta}(1) = 1$, $\xi_{\alpha, \beta}(2) = 0$ and $\xi_{\alpha, \beta}(3) = 3$. So $M(M, G)_{\alpha, \beta} = (-1)^2(1 - t)(1 - t^3)$.

Example 22. We calculate $M_3 = G_3 - (1 - t)G_{21} - (1 - t)G_{12} + (1 - t)(1 - t^2)G_{111}$, $M_{21} = G_{21} - (1 - t)G_{111}$, $M_{12} = G_{12} - (1 - t^2)G_{111}$, and $M_{111} = G_{111}$.

5.3. $M(P, G)$. By multiplying $M(P, F)$ and $M(F, G)$, we obtain the formula

$$(35) \quad M(P, G)_{\lambda, \beta} = \sum_{\substack{S^* = (S, E) \in \text{SYT}^*(\lambda) \\ \text{Asc}'(S^*) \preceq \beta}} (-1)^{|E|} t^{\text{tstat}(S^*) + g(\text{Asc}'(S^*), \beta)}.$$

However, this can be simplified; there is a lot of cancellation. In order to do so, we introduce some new notation.

For $S \in \text{SYT}(\lambda)$, define $\text{Sp}(S)$ and $\text{wt}(c)$ as in §4.1. We define the following subset of $\text{Sp}(S)$:

$$(36) \quad \text{Esp}(S) = \{c \in \text{Sp}(S) : \text{Asc}((S, \{c\})) \neq \text{Asc}((S, \emptyset))\}.$$

So, $c \in \text{Esp}(S)$ if and only if (a) $S(c) - 1$ appears in the column of S just left of the column containing $S(c)$ and (b) $S(c) - 1$ appears weakly lower than $S(c)$ does. The subset $\text{Esp}(S)$ keeps track of which cells actually affect $\text{Asc}'(S^*)$ for starred tableaux with underlying tableau S . As we are only concerned here with standard tableaux S , we will let $c_j = c_j(S)$ denote the cell of S in which j appears. Finally, note that the descent set of S , $\text{Des}(S)$, is contained in $\text{Asc}(S^*)$ for any S^* with underlying tableau S .

Our intent is to derive a simplified version of (35) in which the main sum extends over standard tableaux $S \in \text{SYT}(\lambda)$ rather than starred standard tableaux. To obtain $S^* = (S, E)$ from S , we will build the subset E by choosing to include or exclude various cells $c \in \text{Sp}(S)$. The final object S^* is required to satisfy $\text{Asc}'(S^*) \preceq \beta$, or equivalently, $\text{Asc}(S^*) \subseteq \text{sub}(\beta)$. By the remark at the end of the last paragraph, this requirement will be met only if $\text{Des}(S) \subseteq \text{sub}(\beta)$. So we need only consider standard tableau S satisfying this condition.

The choices of which $c = c_{j+1}(S) \in \text{Sp}(S)$ to include in E can be made independently. We consider each such c according to (a) whether $j \in \text{sub}(\beta)$ and (b) whether $c \in \text{Esp}(S)$. We first note that if $j \notin \text{sub}(\beta)$ and $c \in \text{Esp}(S)$, then inclusion of c in E would cause $\text{Asc}'((S, E)) \not\preceq \beta$. Hence, this case need not be considered further. For each of the remaining possibilities in (a) and (b), we consider the net effect on the signed t -weight caused by the decision to include or exclude c from E . For each $j \in \text{sub}(\beta)$, let $m_j = m_j(\beta)$ be the number of elements in $\text{sub}(\beta)$ that are at most j .

- (1) $j \notin \text{sub}(\beta)$ and $c \notin \text{Esp}(S)$. If c is included in E , the t -weight contribution will be $-t^{\text{wt}(c)}$; otherwise the contribution will be 1.
- (2) $j \in \text{sub}(\beta)$ and $c \in \text{Esp}(S)$. If c is included in E , the t -weight contribution will be $-t^{\text{wt}(c)}$; otherwise it will be t^{m_j} .
- (3) $j \in \text{sub}(\beta)$ and $c \notin \text{Esp}(S)$. Since $c \notin \text{Esp}(S)$, the value of $\xi_{\text{Asc}'(S^*), \beta}(j)$ depends only on whether or not $j \in \text{Des}(S)$. Hence, there is a corresponding t -weight contribution of t^{m_j} if and only if $j \notin \text{Des}(S)$. As in Case 1, there is an additional contribution to the t -weight (coming from the tstat function) of $-t^{\text{wt}(c)}$ or 1 according to whether or not c is in E .

Observing that the factor $1 - t^{\text{wt}(c)}$ occurs in both Case 1 and Case 3, we reorganize the cases slightly to obtain the following.

Theorem 23. *For all $\lambda \in \text{Par}_n$ and $\beta \in \text{Comp}_n$,*

$$(37) \quad M(P, G)_{\lambda, \beta} = \sum_{\substack{S \in \text{SYT}(\lambda) \\ \text{Des}(S) \subseteq \text{sub}(\beta)}} \prod_{\substack{j \in \text{sub}(\beta): \\ c_{j+1} \in \text{Esp}(S)}} (t^{m_j} - t^{\text{wt}(c_{j+1})}) \prod_{j: c_{j+1} \in \text{Sp}(S) \setminus \text{Esp}(S)} t^{m'_j} (1 - t^{\text{wt}(c_{j+1})}),$$

where $m'_j = m_j$ if $j \in \text{sub}(\beta) \setminus \text{Des}(S)$ and 0 otherwise.

Corollary 24. *If $m_j = \text{wt}(c_{j+1})$ for some $j \in \text{sub}(\beta)$ with $c_{j+1} \in \text{Esp}(S)$, then S can be omitted from the sum in (37).*

Example 25. Let $\lambda = 32$ and $\beta = 1211$. Note that $\text{sub}(\beta) = \{1, 3, 4\}$ and so $m_1 = 1$, $m_3 = 2$, and $m_4 = 3$. In Table 1 we list the five elements of $\text{SYT}(32)$ (referred to from left to right as S_1, \dots, S_5) along with pertinent data. The row labeled \prod_1 (resp. \prod_2) gives the contributions from the first (resp. second) product in (37). Since $\text{Des}(S_1), \text{Des}(S_2) \not\subseteq \text{sub}(\beta)$, \prod_1 and \prod_2 have been left blank for these two tableaux. (For reference, the corresponding products for these tableaux are $(t - t) \cdot t^2(1 - t)(1 - t)$ and $(t - t)(t^2 - t)(t^3 - t^2) \cdot 1$, respectively.) Note that Corollary 24 applies to S_1 with $j = m_j = 1$. So the only contributions are from the last two columns and we find that

$$(38) \quad M(P, G)_{32, 1211} = (t^2 - t)(1 - t) + (t^3 - t^2)(1 - t) = -t^4 + t^3 + t^2 - t.$$

TABLE 1. Computation of $M(P, G)_{32,1211}$.

S	<table><tr><td>1</td><td>2</td><td>3</td></tr><tr><td>4</td><td>5</td><td></td></tr></table>	1	2	3	4	5		<table><tr><td>1</td><td>2</td><td>4</td></tr><tr><td>3</td><td>5</td><td></td></tr></table>	1	2	4	3	5		<table><tr><td>1</td><td>2</td><td>5</td></tr><tr><td>3</td><td>4</td><td></td></tr></table>	1	2	5	3	4		<table><tr><td>1</td><td>3</td><td>4</td></tr><tr><td>2</td><td>5</td><td></td></tr></table>	1	3	4	2	5		<table><tr><td>1</td><td>3</td><td>5</td></tr><tr><td>2</td><td>4</td><td></td></tr></table>	1	3	5	2	4	
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$\text{Des}(S)$	$\{3\}$	$\{2, 4\}$	$\{2\}$	$\{1, 4\}$	$\{1, 3\}$																														
$\text{Sp}(S)$	$\{c_2, c_3, c_5\}$	$\{c_2, c_4, c_5\}$	$\{c_2, c_4, c_5\}$	$\{c_3, c_4, c_5\}$	$\{c_3, c_4, c_5\}$																														
$\text{Esp}(S)$	$\{c_2, c_3, c_5\}$	$\{c_2\}$	$\{c_2, c_4, c_5\}$	$\{c_3, c_4\}$	$\{c_3, c_5\}$																														
\prod_1	$(t-t)(t^3-t)$			(t^2-t)	(t^3-t^2)																														
\prod_2	1			$(1-t)$	$(1-t)$																														

6. APPENDIX: EXAMPLES OF TRANSITION MATRICES

This appendix lists specific examples of transition matrices (old and new) discussed in this paper. In each case, we give the relevant matrix for $n = 4$.

$$M(s, m) = \begin{matrix} & 4 & 31 & 22 & 211 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(s, F) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(m, M) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(P, m) = \begin{matrix} & 4 & 31 & 22 & 211 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & 1-t & 1-t & (1-t)^2 & (1-t)^3 \\ 0 & 1 & 1-t & 2(1-t) & 3-5t+t^2+t^3 \\ 0 & 0 & 1 & 1-t & 2-3t+t^3 \\ 0 & 0 & 0 & 1 & 3-t-t^2-t^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(m, s) = \begin{matrix} & 4 & 31 & 22 & 211 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(s, P) = \begin{matrix} & 4 & 31 & 22 & 211 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & t & t^2 & t^3 & t^6 \\ 0 & 1 & t & t+t^2 & t^3+t^4+t^5 \\ 0 & 0 & 1 & t & t^2+t^4 \\ 0 & 0 & 0 & 1 & t+t^2+t^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(P, s) = \begin{matrix} & 4 & 31 & 22 & 211 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & -t & 0 & t^2 & -t^3 \\ 0 & 1 & -t & -t & t^2+t^3 \\ 0 & 0 & 1 & -t & t^3 \\ 0 & 0 & 0 & 1 & -t-t^2-t^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(F, M) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(M, F) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(G, F) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & -t & -t & t^2 & -t & t^2 & t^2 & -t^3 \\ 0 & 1 & 0 & -t & 0 & -t & 0 & t^2 \\ 0 & 0 & 1 & -t^2 & 0 & 0 & -t & t^3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 0 & 1 & -t^2 & -t^2 & t^4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(G, M) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & -t+1 & -t+1 & t^2-2t+1 & -t+1 & t^2-2t+1 & t^2-2t+1 & -t^3+3t^2-3t+1 \\ 0 & 1 & 0 & -t+1 & 0 & -t+1 & 0 & t^2-2t+1 \\ 0 & 0 & 1 & -t^2+1 & 0 & 0 & -t+1 & t^3-t^2-t+1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -t+1 \\ 0 & 0 & 0 & 0 & 1 & -t^2+1 & -t^2+1 & t^4-2t^2+1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -t^2+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -t^3+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(P, F) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & -t & -t & t^2 & -t & t^2 & t^2 & -t^3 \\ 0 & 1 & -t+1 & -t & 1 & -2t & -t & t^3+t^2 \\ 0 & 0 & 1 & -t & 0 & -t+1 & -t & t^3 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & -t^3-t^2-t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(F, G) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & t & t & t^3 & t & t^3 & t^3 & t^6 \\ 0 & 1 & 0 & t & 0 & t & 0 & t^3 \\ 0 & 0 & 1 & t^2 & 0 & 0 & t & t^4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & t^2 & t^2 & t^5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(M, G) =$$

$$\begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & t-1 & t-1 & t^3-t^2-t+1 & t-1 & t^3-t^2-t+1 & t^3-t^2-t+1 & t^6-t^5-t^4+t^2+t-1 \\ 0 & 1 & 0 & t-1 & 0 & t-1 & 0 & t^3-t^2-t+1 \\ 0 & 0 & 1 & t^2-1 & 0 & 0 & t-1 & t^4-t^3-t+1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & t-1 \\ 0 & 0 & 0 & 0 & 1 & t^2-1 & t^2-1 & t^5-t^3-t^2+1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & t^2-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & t^3-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$M(P, G) = \begin{matrix} & 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{matrix} 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -t+1 & -t^3+t^2 & 1 & t^2-t & 0 & 0 \\ 0 & 0 & 1 & t^2-t & 0 & -t+1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

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